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SLANT SUB-MANIFOLDS OF GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract. In this paper, we study the slant submanifolds of generalized Sasakian space forms when structure tensor field ϕ is Killing. Also, obtained the conditions for anti-invariant submanifolds under some geometrical conditions such as $\nabla Q = 0$ and $\nabla T = 0$.

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1. Introduction

Slant immersions in complex geometry were defined by Chen [3] as a natural generalization of both holomorphic immersions and totally real immersions. In [4], Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold and he has proved some properties of such immersions. Lotta [6] has obtained examples of slant sub-manifolds in the Sasakian-space-form $\mathbb{R}^{2n+1}$ as the leaves of a harmonic Riemannian 3-dimensional foliation. Finally, Lotta [5] has also studied some properties about the intrinsic geometry of 3-dimensional non-anti-invariant slant sub-manifolds of K-contact manifolds.

Alegre et al. [1] introduced and studied the generalized Sasakian-space-forms. The authors Alegre and Carriazo [2], Somashekhara and Nagaraja [7] and Nagaraja et al. [9] studied the generalized Sasakian-space-forms.

In [8], the authors studied the invariant and anti-invariant submanifolds of (κ, μ)-contact metric manifolds as Ricci solitons. In [10, 11], the authors studied some results on indefinite Sasakian manifold admitting quarter-symmetric metric connection and η-Ricci solitons of some curvature tensors and investigated certain curvature tensor on indefinite trans-Sasakian manifold.

2. Preliminaries

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors ϕ, ξ, η, g, where ϕ is a $(1,1)$ tensor field, ξ is a vector field,

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η is a 1-form and g the Riemannian metric on $\overline{M}$. We have

$$\phi^2(X_1) = -X_1 + \eta(X_1)\xi, \;\; \phi\xi = 0, \;\; \eta(\phi X_1) = 0, \tag{1}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \;\; g(X_1, \xi) = \eta(X_1). \tag{2}$$

For any $X_1, X_2 \in T\overline{M}$, where $T\overline{M}$ denote the Lie algebra of vector fields on $\overline{M}$. An almost contact metric manifold is called a generalized Sasakian-space-form [1, 12] if

$$\begin{aligned} \overline{R}(X_1, X_2)X_3 = f_1\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\} \\ +f_2\{g(X_1, \phi X_3)\phi X_2 - g(X_2, \phi X_3)\phi X_1 + 2g(X_1, \phi X_2)\phi X_3\} \\ +f_3\{\eta(X_1)\eta(X_3)X_2 - \eta(X_2)\eta(X_3)X_1 \\ +g(X_1, X_3)\eta(X_2)\xi - g(X_2, X_3)\eta(X_1)\xi\} \end{aligned} \tag{3}$$

The following hold in a generalized Sasakian space form.

$$(\overline{\nabla}_{X_1}\phi)(X_2) = (f_1 - f_3)(g(X_1, X_2)\xi - \eta(X_2)X_1), \tag{4}$$

$$\overline{\nabla}_{X_1}\xi = -(f_1 - f_3)\phi X_1, \tag{5}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. We denote by TM the Lie algebra of vector fields on M and by $T^{\perp}M$ the set of all vector normal fields to M. For any $X_1 \in TM$ and $N \in T^{\perp}M$, we have:

$$\phi X_1 = TX_1 + \omega X_1, \quad \phi N = BN + CN, \tag{6}$$

where TX_1 denotes the tangential component of ϕX_1 and BN denotes the tangential component of ϕN. The structure vector field ξ is tangent to M. Further, if we denote D by the orthogonal distribution to ξ in TM, then we can take the orthogonal direct decomposition $TM = D \perp \xi$. X_1 is not proportional to ξ_x, for each non-zero X_1 tangent to M at x.

The Wirtinger angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle θ equal to 0 and $\frac{\pi}{2}$ respectively. A slant immersion which is not invariant nor anti-invariant is called a proper slant immersion.

Let $\overline{\nabla}$ be the Riemannian connection on M. Formulae of Gauss and Weingarten are given by

$$\overline{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + h(X_1, X_2), \tag{7}$$

$$\overline{\nabla}_{X_1}N = -A_N X_1 + \nabla^{\perp}_{X_1}N, \tag{8}$$

for $X_1, X_2 \in TM$ and $N \in T^{\perp}M$ of ∇M; h and A_N are the second fundamental forms related by

$$g(A_N X_1, X_2) = g(h(X_1, X_2), N). \tag{9}$$

The mean curvature vector H is defined by $H = m^{-1}$ trace h. If T is an endomorphism defined by (6) then

$$g(TX_1, X_2) + g(X_1, TX_2) = 0. \tag{10}$$

Thus T^2, can be referred by Q, which is self-adjoint. Use (4) and (6) in the Gauss and Weingarten formulae, then we have:

$$(\nabla_{X_1} T)X_2 = (f_1 - f_3)[g(X_1, X_2)\xi - \eta(X_2)X_1] + A_{\omega X_2}X_1 + Bh(X_1, X_2), \tag{11}$$

$$(\nabla_{X_1}\omega)X_2 = Ch(X_1, X_2) - h(X_1, TX_2), \tag{12}$$

where B and C denotes the tangential and normal components of ϕN.

A tensor field ϕ is said to be Killing if

$$(\overline{\nabla}_{X_1}\phi)X_2 + (\overline{\nabla}_{X_2}\phi)X_1 = 0. \tag{13}$$

3. Slant Sub-manifolds of Generalized Sasakian-space-forms with Killing structure tensor field ϕ

In this section, we consider the slant sub-manifolds of generalized Sasakian-space-forms with Killing structure tensor field ϕ of type $(1,1)$. If M is a sub-manifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$, then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$T^2X_1 = -\lambda(X_1 - \eta(X_1)\xi). \tag{14}$$

If θ is the angle of M, then $\lambda = cos^2\theta$.

Theorem 3.1. *Let M be a 3-dimensional sub-manifold of a generalized Sasakian-space-form with Killing structure tensor field ϕ. Then M is slant if and only if*

$$\eta(X_2)TX_1 + \eta(X_1)TX_2 = 0, \tag{15}$$

and

$$\eta(X_2)\omega X_1 + \eta(X_1)\omega X_2 = 0, \tag{16}$$

provided $(f_1 - f_3) \neq 0$.

Proof. By using (4), we have

$$(\overline{\nabla}_{X_1}\phi)X_2 + (\overline{\nabla}_{X_2}\phi)X_1 = (f_1 - f_3)[2g(X_1, X_2)\xi - (\eta(X_1)X_2 + \eta(X_2)X_1)]. \tag{17}$$

By using (13), we have

$$(f_1 - f_3)[2g(X_1, X_2)\xi - (\eta(X_1)X_2 + \eta(X_2)X_1)] = 0. \tag{18}$$

Apply ϕ on both the sides and using (6), we get

$$-(f_1 - f_3)[\eta(X_1)(TX_2 + \omega X_2) + \eta(X_2)(TX_1 + \omega X_1)] = 0. \tag{19}$$

Equating tangential and normal part to zero, we get (15) and (16), provided $f_1 - f_3 \neq 0$.
The converse part follows by the straightforward computation. □

Lemma 3.2. *Let M be a 3-dimensional sub-manifold of a generalized Sasakian-space-form with Killing structure tensor field ϕ. Then*

$$2g(X_1, X_2)\xi + A_{\omega X_2}X_1 + A_{\omega X_1}X_2 + 2Bh(X_1, X_2) = 0, \tag{20}$$

provided $(f_1 - f_3) \neq 0$ for any $X_1, X_2 \in TM$.

Proof. From the equation (11), we get

$$(\nabla_{X_1}T)X_2 = (f_1 - f_3)[g(X_1, X_2)\xi - \eta(X_2)X_1] + A_{\omega X_2}X_1 + Bh(X_1, X_2). \tag{21}$$

Interchanging X_1 to X_2 in the above equation, we have

$$(\nabla_{X_2}T)X_1 = (f_1 - f_3)[g(X_1, X_2)\xi - \eta(X_1)X_2] + A_{\omega X_1}X_2 + Bh(X_1, X_2). \tag{22}$$

On combining (21) and (22), we have

$$\begin{aligned}(\nabla_{X_1}T)X_2 + (\nabla_{X_2}T)X_1 &= (f_1 - f_3)[2g(X_1, X_2)\xi - \eta(X_2)X_1 - \eta(X_1)X_2] \\ &\quad + A_{\omega X_2}X_1 + A_{\omega X_1}X_2 + 2Bh(X_1, X_2),\end{aligned} \tag{23}$$

In view of (15),(16) and also by the definition of Killing structure we obtain (20).
Hence we have

$$2g(X_1, X_2)\xi + A_{\omega X_2}X_1 + A_{\omega X_1}X_2 + 2Bh(X_1, X_2) = 0, \tag{24}$$

provided $(f_1 - f_3) \neq 0$. □

Theorem 3.3. *Let M be a 3-dimensional sub-manifold of a generalized Sasakian-space-form $\overline{M}$ then*

$$(\overline{\nabla}_{X_1}\omega)X_2 + (\overline{\nabla}_{X_2}\omega)X_1 = 0, \tag{25}$$

if and only if

$$2Ch(X_1, X_2) = h(X_1, TX_2) + h(X_2, TX_1). \tag{26}$$

Proof. Interchanging X_1 and X_2 in (12), we have

$$(\nabla_{X_2}\omega)X_1 = Ch(X_1, X_2) - h(X_2, TX_1). \tag{27}$$

Adding (12) and (27), we obtain

$$(\nabla_{X_1}\omega)X_2 + (\nabla_{X_2}\omega)X_1 = 2Ch(X_1, X_2) - h(X_2, TX_1) - h(X_1, TX_2). \tag{28}$$

Now by using (16) in (28), we obtain

$$2Ch(X_1, X_2) = h(X_2, TX_1) + h(X_1, TX_2), \tag{29}$$

which proves the result. □

Lemma 3.4. *Let M be a slant sub-manifold of a generalized Sasakian-space-form $\overline{M}$. Then we have*

$$h(X_1, \xi) = -(f_1 - f_3)\omega X_1 \text{ and } \nabla_{X_1}\xi = -(f_1 - f_3)TX_1. \tag{30}$$

Proof. Put $X_2 = \xi$ in (7), we get

$$\overline{\nabla}_{X_1}\xi = \nabla_{X_1}\xi + h(X_1, \xi).$$

Taking (5) in the above equation, we obtain

$$-(f_1 - f_3)\phi X_1 = \nabla_{X_1}\xi + h(X_1, \xi).$$

Now using (6) in the above equation and on comparing tangential and normal part, we have

$$h(X_1, \xi) = -(f_1 - f_3)\omega X_1 \textit{ and } \nabla_{X_1}\xi = -(f_1 - f_3)TX_1,$$

which proves (30). □

Theorem 3.5. *Let M be a sub-manifold of a generalized Sasakian-space-form $\overline{M}$. The endomorphism T is parallel if and only if M is anti-invariant.*

Proof. If M is an anti-invariant sub-manifold of $\overline{M}$, then $T = 0$ and so $\nabla T = 0$.
Conversly, if $\nabla T = 0$ then by (11), we obtain

$$(f_1 - f_3)[g(X_1, X_2)\xi - \eta(X_2)X_1] + g(A_{\omega X_2}X_1, \xi) = 0, \tag{31}$$

for any $X_1, X_2 \in (TM)$. Using the (9), we get

$$(f_1 - f_3)[g(X_1, X_2)\xi - \eta(X_2)X_1] + g(h(X_1, \xi), \omega X_2) = 0.$$

Therefore, by using (30) in the above equation, we get $(f_1-f_3)[g(TX_1, TX_2)] = 0$ provided $f_1 - f_3 \neq 0$ which implies that $T = 0$, that is M is an anti-invariant. □

Theorem 3.6. *Let M be a slant sub-manifold of a generalized Sasakian-space-form $\overline{M}$ such that $\xi \in TM$. Then $\nabla Q = 0$ if and only if M is anti-invariant sub-manifold.*

Proof. From the equation (14), considering $T^2 = Q$, we get:

$$QX_2 = \lambda(-X_2 + \eta(X_2))\xi. \tag{32}$$

Now taking the covariant differentiation of the above equation and by using (5), we get

$$(\nabla_{X_1}Q)X_2 = \lambda[-\eta(X_2)(f_1 - f_3)\phi X_1 - g(\nabla_{X_1}X_2, \xi)\xi]. \tag{33}$$

Now by using

$$g(\nabla_{X_1}X_2, \xi) = \nabla_{X_1}\eta(X_2) + g(X_2, \phi X_1)(f_1 - f_3)$$

in (33), we obtain

$$(\nabla_{X_1}Q)X_2 = \lambda(f_1 - f_3)g(\phi X_2, X_1)\xi. \tag{34}$$

Now from (6), we have

$$(\nabla_{X_1}Q)X_2 = \lambda(f_1 - f_3)g(TX_2, X_1)\xi. \tag{35}$$

Here, we note that

$$(f_1 - f_3)g(TX_2, X_1)\xi \neq 0 \tag{36}$$

Hence $\nabla Q = 0$ if and only if $\theta = \frac{\pi}{2}$, which proves our assertion. □

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